Last-Iterate Convergence of Optimistic Gradient Method for Monotone Variational Inequalities

Eduard Gorbunov[1,2,3], Adrien Taylor[4], Gauthier Gidel[2,5]

[1]MIPT, [2]Mila & UdeM, [3]MBZUAI,
[4]INRIA & ENS & CNRS & PSL,
[5]Canada CIFAR AI Chair

1. Preliminaries

Problem: variational inequality problem (VIP) – find $x^* \in \mathcal{X} \subseteq \mathbb{R}^d$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{X}$$

Examples:

- Min-max problems $\quad \min_{u \in U} \max_{v \in V} f(u,v)$
- Minimization problems $\quad \min_{x \in \mathcal{X}} f(x)$

Assumptions: for all $x, y \in \mathcal{X}$ we assume

- Lipschitzness $\quad \|F(x) - F(y)\| \leq L\|x - y\|$
- Monotonicity $\quad \langle F(x) - F(y), x - y \rangle \geq 0$

Convergence metrics:

- Restricted gap function: for $R \sim \|x^0 - x^*\|$ it is defined as

$$\mathrm{Gap}_F(x^N) = \max_{y \in \mathcal{X}: \|y - x^*\| \leq R} \langle F(y), x^N - y \rangle$$

- Squared norm of the residual/operator (constrained/unconstrained cases):

$$\|x^N - x^{N-1}\|^2 \qquad \|F(x^N)\|^2$$

By default we report all results in terms of the squared norm of the residual

2. Extragradient and Past Extragradient

Extragradient method (EG) [Korpelevich, 1976]

$$\widetilde{x}^k = \mathrm{proj}\left[x^k - \gamma F(x^k)\right], \quad x^{k+1} = \mathrm{proj}\left[x^k - \gamma F(\widetilde{x}^k)\right]$$

- $\mathrm{proj}[x] = \arg\min_{y \in \mathcal{X}} \|y - x\|$ – projection operator

Past Extragradient/Optimistic Gradient method (PEG) [Popov, 1980]

$$\widetilde{x}^k = \mathrm{proj}\left[x^k - \gamma F(\widetilde{x}^{k-1})\right], \quad x^{k+1} = \mathrm{proj}\left[x^k - \gamma F(\widetilde{x}^k)\right]$$

In contrast to EG, PEG

- Requires only 1 operator call per iteration
- Is implementable as no-regret algorithm

Last-iterate convergence results for

EG

- $\mathcal{O}(1/N)$ bound in the unconstrained case [Gorbunov et al., 2022]
- $\mathcal{O}(1/N)$ bound in the constrained case [Cai et al., 2022]

PEG

- $\mathcal{O}(1/N)$ bound in the unconstrained case *if additionally the Jacobian $\nabla F(x)$ is Λ-Lipschitz* [Golowich et al., 2020]

3. Our Contributions

$\mathcal{O}(1/N)$ last-iterate convergence rate for PEG in terms of the squared norm of the residual for monotone and Lipschitz VIPs in constrained and unconstrained cases

✓ No additional assumptions are used ✓ Potential-based proof obtained via computer

4. Main Results

Unconstrained case

Key lemma: for any $k > 0$ the iterates of **PEG** satisfy

$$\Psi_{k+1} \leq \Psi_k - 3\left(\frac{2}{9} - L^2\gamma^2\right)\left\|F(\widetilde{x}^k) - F(\widetilde{x}^{k-1})\right\|^2$$

for $\Psi_k = \left\|F(x^k)\right\|^2 + 2\left\|F(x^k) - F(\widetilde{x}^{k-1})\right\|^2$

- In contrast, **EG** has much simpler potential: $\Psi_k = \|F(x^k)\|^2$
- As we illustrate in the paper, $\|F(x^k)\|^2$ is not a potential for **PEG**, i.e., $\|F(x^k)\|^2$ can grow for **PEG**

Using this lemma and standard analysis of **PEG**, we derive the following result

Theorem: for any $k > 0$ the iterates of **PEG** with $\gamma \leq 1/3L$ satisfy

$$\Phi_{k+1} \leq \Phi_k, \quad \Phi_k = \left\|x^k - x^*\right\|^2 + \frac{k+32}{3}\gamma^2\Psi_k$$

In particular, this implies

$$\left\|F(x^N)\right\|^2 = \mathcal{O}\left(\frac{R_0^2}{\gamma^2 N}\right) \quad \mathrm{Gap}_F(x^N) = \mathcal{O}\left(\frac{R_0}{\gamma\sqrt{N}}\right)$$

Constrained case

Theorem: for any $k > 1$ the iterates of **PEG** with $\gamma \leq 1/4L$ satisfy

$$\Phi_{k+1} \leq \Phi_k$$

$$\Phi_k = \left\|x^k - x^*\right\|^2 + \frac{1}{16}\left\|\widetilde{x}^{k-1} - \widetilde{x}^{k-2}\right\|^2 + \frac{3k+32}{24}\Psi_k$$

$$\Psi_k = \left\|x^k - x^{k-1}\right\|^2 + \left\|x^k - x^{k-1} - 2\gamma\left(F(x^k) - F(\widetilde{x}^{k-1})\right)\right\|^2$$

In particular, this implies

$$\left\|x^N - x^{N-1}\right\|^2 = \mathcal{O}\left(\frac{R_0^2}{N}\right) \quad \mathrm{Gap}_F(x^N) = \mathcal{O}\left(\frac{R_0}{\gamma\sqrt{N}}\right)$$

- Analysis does not follow straightforwardly from the result in the unconstrained case

5. Path to the Proof

Below we illustrate the non-triviality of the analysis of **PEG** even for unconstrained VIPs. To obtain the results below and the proofs most of the results in the paper we used Performance Estimation Problems technique [Taylor et al., 2017].

- The following problem gives the worst-case last-iterate guarantee for **PEG**

$$G_{\mathrm{PEG}}(\gamma, L, N) = \max_{\substack{F, d, x^* \\ \tilde{x}^0, \ldots, \tilde{x}^N \\ x^0, \ldots, x^N}} \frac{\|F(x^N)\|^2}{\|x^0 - x^*\|^2}$$

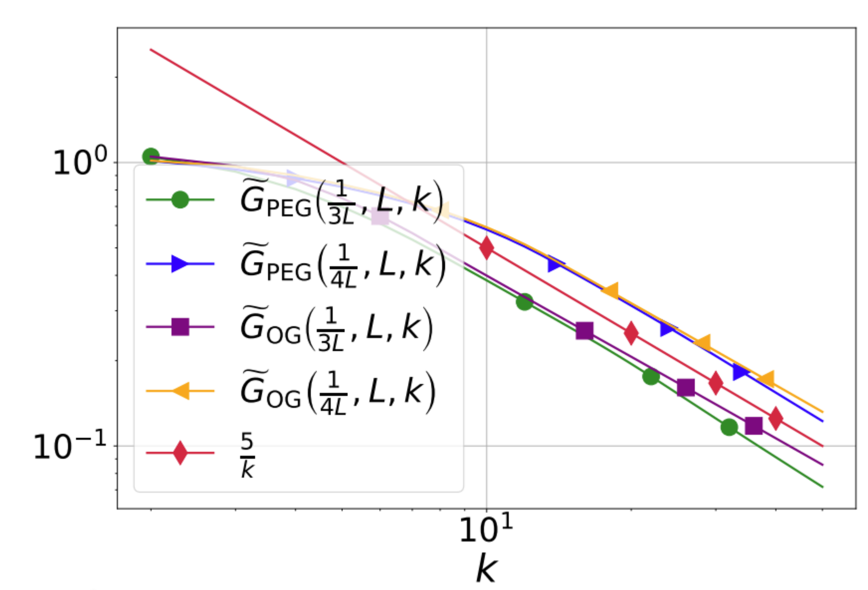
$$\text{s.t.} \quad \begin{aligned} &F \text{ is monotone and } L\text{-Lipschitz}, \\ &\tilde{x}^0 = x^0 \in \mathbb{R}^d, x^1 = x^0 - \gamma F(x^0) \\ &\widetilde{x}^k = x^k - \gamma F(\widetilde{x}^{k-1}), \text{ for } k = 1, \ldots, N, \\ &x^{k+1} = x^k - \gamma F(\widetilde{x}^k), \text{ for } k = 1, \ldots, N-1 \end{aligned}$$

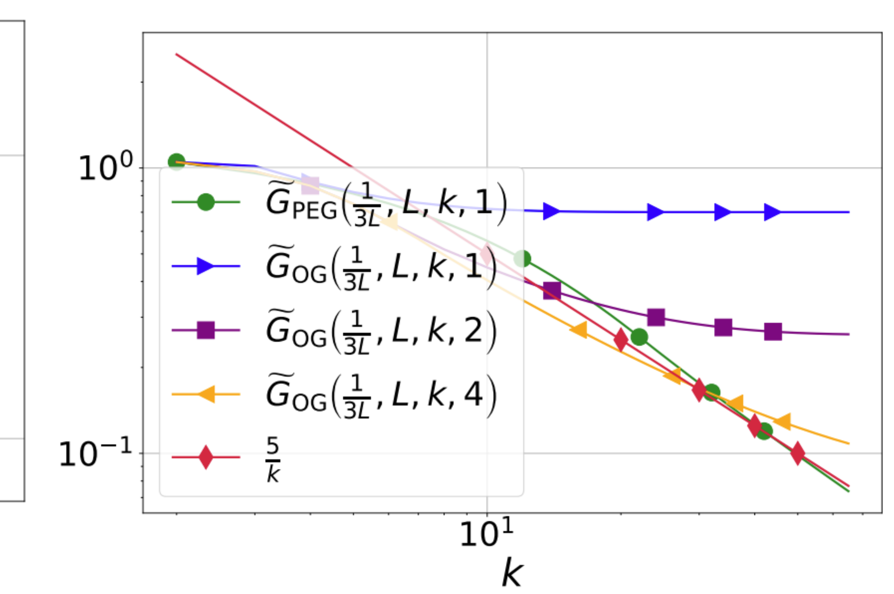
- Bad news: problem is infinitely-dimensional→ hard to solve
- Good news: there exist an SDP relaxation that is easy to solve numerically
- SDP finds pairs $\{x^*, 0\}$, $\{x^k, g^k\}_{k=0}^N$, $\{\tilde{x}^k, \tilde{g}^k\}_{k=0}^N$ such that $g^k \approx F(x^k)$, $\tilde{g}^k \approx F(\tilde{x}^k)$ and all Lipschitzness and monotonicity inequalities between these points hold
- This SDP has optimal value $\tilde{G}_{\mathrm{PEG}}(\gamma, L, N) \geq G_{\mathrm{PEG}}(\gamma, L, N)$. We numerically verified that $\tilde{G}_{\mathrm{PEG}}(\gamma, L, N) = \mathcal{O}(1/N)$

In the unconstrained case, one can rewrite **PEG** in the Optimistic Gradient (**OG**) form

$$\widetilde{x}^{k+1} = \widetilde{x}^k - 2\gamma F(\widetilde{x}^k) + \gamma F(\widetilde{x}^{k-1})$$

- Does not use sequence $\{x^k\}_{k\geq 0}$, looks simpler
- Similarly to **PEG**, one can formulate SDP for **OG** and verify $\tilde{G}_{\mathrm{OG}}(\gamma, L, N) = \mathcal{O}(1/N)$
- However, it is hard to find a simple proof for **OG:** consider SDPs $\tilde{G}_{\mathrm{PEG}}(\gamma, L, N, t)$ and $\tilde{G}_{\mathrm{OG}}(\gamma, L, N, t)$ obtained from $\tilde{G}_{\mathrm{PEG}}(\gamma, L, N)$ and $\tilde{G}_{\mathrm{OG}}(\gamma, L, N)$ via removing the constraints corresponding to the points from steps i, j such that $|i - j| > t$. While $\tilde{G}_{\mathrm{PEG}}(\gamma, L, N, t) = \mathcal{O}(1/N)$, $\tilde{G}_{\mathrm{OG}}(\gamma, L, N, t)$ does not even for $t = 4$.





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